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Optimal Recovery by Linear Functionals

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Micchelli [1] presented the following problem: Let X, Y, Z be linear spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let Y, Z be endowed with norms, both denoted by $\|\cdot\|$. Let

$$I: X \to Y$$
$$U: X \to Z$$

be linear mappings. I is the so-called "information mapping" and U the mapping we want to "recover." Furthermore, a set $K \subseteq X$ and an information error $\varepsilon \geqslant 0$ are given. An algorithm is a mapping

$$A: B(IK, \varepsilon] \to Z.$$

Here

$$B(IK, \varepsilon] := \{y \in Y;\ \|Ix - y\| \leqslant \varepsilon \text{ for some } x \in K\}.$$

The error of an algorithm A is defined as

$$E_A(K, \varepsilon) := \sup\{\|Ux - Ay\|;\ \|Ix - y\| \leqslant \varepsilon,\ x \in K\}.$$

An algorithm A_0 satisfying

$$E_{A_0} = E(K, \varepsilon) := \inf_A E_A(K, \varepsilon)$$

is called an optimal algorithm.

We are making the restriction $Z = \mathbb{K} = \mathbb{R}$ and assume K to be convex and balanced, e.g., $x \in K$ implies $-x \in K$. We assume $E(K, \varepsilon) < \infty$, since otherwise everything is trivial. We write $\sup M$ ($\inf M$) for the supremum (infinum) of a set $M \subseteq \mathbb{R}$.

Our result is given in the following theorem:

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THEOREM. *There is always a linear optimal algorithm. There is a linear and continuous algorithm, if one of the following conditions hold:*

(a) $\varepsilon > 0$,

(b) *IK is a neighborhood of* 0 *and there is a linear and continuous algorithm L with $E_L(K, \varepsilon) < \infty$.*

There is only one linear optimal algorithm, iff IK is absorbing and for all $y \in Y$,

$$\lim_{\lambda \to 0_+} \frac{\Phi_\varepsilon(\lambda y) - \Phi_\varepsilon(0)}{\lambda} = \lim_{\lambda \to 0_-} \frac{\Phi_\varepsilon(\lambda y) - \Phi_\varepsilon(0)}{\lambda}, \tag{1}$$

with

$$\Phi_\varepsilon(y) := \sup U(I^{-1}(B(y, \varepsilon]) \cap K).$$

Case (a) was proved by Micchelli [1] and Micchelli and Rivlin [2]. Also the condition for uniqueness has been given in the same paper. For completeness we give a complete and alternative proof of the whole theorem.

We use the following notations:

$$\begin{aligned} Y^* &:= \{L; L\colon Y \to \mathbb{R} \text{ linear}\}, \\ Y' &:= \{L; L\colon Y \to \mathbb{R} \text{ continuous and linear}\}, \\ B(y, \varepsilon] &:= \{x \in Y; \|x - y\| \leqslant \varepsilon\}, \\ H_\varepsilon(y) &:= U(I^{-1}(B(y, \varepsilon]) \cap K\}. \end{aligned}$$

It is clear that $H_\varepsilon(y) = \varnothing$ iff $y \notin B(IK, \varepsilon]$. Thus Φ_ε is defined on the convex set $B(IK, \varepsilon]$. Φ_ε is a concave function. As in [2] we define

$$\begin{aligned} e(K, \varepsilon) &:= \sup\{|Ux|; \|Ix\| \leqslant \varepsilon, x \in K\} \\ &= \Phi_\varepsilon(0). \end{aligned}$$

For later use we remark

$$\begin{aligned} \Phi_\varepsilon(-y) &= \sup\{Ux; Ix \in B(y, \varepsilon], x \in K\} \\ &= -\inf\{Ux; Ix \in B(y, \varepsilon], x \in K\} \\ &= -\inf H_\varepsilon(y) \end{aligned}$$

since K is balanced.

Thus

$$\overline{H_\varepsilon(y)} = [-\Phi_\varepsilon(-y), \Phi_\varepsilon(y)].$$

$H_\varepsilon(y)$ is bounded, since we assume $E(K, \varepsilon) < \infty$. In [2] it is proved that

$$e(K, \varepsilon) = E(K, \varepsilon). \tag{2}$$

We now give the basic lemma:

LEMMA 1. *$L \in Y^*$ is optimal iff*

$$L(y) \geqslant \Phi_\varepsilon(y) - \Phi_\varepsilon(0) \tag{3}$$

for all $y \in B(IK, \varepsilon]$.

Proof. (1) Let $L \in Y^*$ be optimal, but

$$\Phi_\varepsilon(y) - L(y) > \Phi_\varepsilon(0) = e(K, \varepsilon).$$

We get

$$E(K, \varepsilon) = E_L(K, \varepsilon) \geqslant \Phi_\varepsilon(y) - L(y) > e(K, \varepsilon).$$

This is a contradiction to (2).

(2) Suppose (3) holds. Then for all $y \in B(IK, \varepsilon]$

$$\Phi_\varepsilon(y) - L(y) \leqslant \Phi_\varepsilon(0) = e(K, \varepsilon).$$

Since K is balanced we get $H_\varepsilon(y) = -H_\varepsilon(-y)$ and

$$\inf(H_\varepsilon(y)) = -\sup(H_\varepsilon(-y)) = -\Phi_\varepsilon(-y).$$

Thus

$$\begin{aligned} L(y) - \inf(H_\varepsilon(y)) &= L(y) + \Phi_\varepsilon(-y) \\ &= \Phi_\varepsilon(-y) - L(-y) \\ &\leqslant \Phi_\varepsilon(0) = e(K, \varepsilon). \end{aligned}$$

We now have

$$|L(y) - z| \leqslant e(K, \varepsilon)$$

for all $z \in H_\varepsilon(y)$. Thus

$$E_L(K, \varepsilon) \leqslant e(K, \varepsilon) = E(K, \varepsilon). \qquad \text{Q.E.D.}$$

Now we have to find all $L \in Y^*$ with

$$L(y) \geqslant \Phi_\varepsilon(y) - \Phi_\varepsilon(0)$$

for all $y \in B(IK, \varepsilon]$. But the existence of such a $L \in Y^*$ is a consequence of the following lemma.

LEMMA 2. *Let E be a linear $\mathbb{R}$-space, $U \subseteq E$ a convex and balanced set. Then for any concave functional $\varphi: U \to \mathbb{R}$, $\varphi(0) = 0$, there is a $L \in E^*$ with*

$$L(y) \geqslant \varphi(y)$$

for all $y \in U$.

Proof. Let F be the linear subspace spanned by U. Since U is balanced and convex, U is absorbing in F. Since φ is concave it is differentiable into any direction $y \in F$, e.g.,

$$h(y) = \lim_{\lambda \to 0_+} \frac{\varphi(\lambda y)}{\lambda} = \sup_{\lambda \to 0_+} \frac{\varphi(\lambda y)}{\lambda}. \tag{4}$$

is well defined in F. $-h$ is a sublinear functional on F and by the Hahn–Banach theorem there is a functional $\tilde{L} \in F^*$ with

$$\tilde{L}(y) \leqslant -h(y) \leqslant -\varphi(y)$$

for all $y \in U$.

Thus there is a functional $L \in E^*$ with the desired property.

It is now proved that there is always a linear optimal algorithm. A closer look to the proof of Lemma 2 shows that condition (1) from the theorem is necessary and sufficient for the uniqueness of the optimal linear algorithm. In this case the sublinear functional h in (4) is the optimal algorithm in Y^*.

By (3) in Lemma 1 it is clear that any linear optimal algorithm is bounded, if Φ_ε is bounded in a neighborhood of 0. We examine the two cases, (a) and (b), of the theorem:

(a) $\varepsilon > 0$: Now $B(IK, \varepsilon]$ is a neighborhood of 0. Let $\|y\| \leqslant \varepsilon$, $\|Ix - y\| \leqslant \varepsilon$, $x \in K$.

Then

$$2\,\|I(\tfrac{1}{2}x)\| = \|I(x)\| \leqslant 2\varepsilon.$$

Thus

$$\|I(\tfrac{1}{2}x)\| \leqslant \varepsilon.$$

We have now

$$|U(\tfrac{1}{2}x)| \leqslant \Phi_\varepsilon(0).$$

This implies

$$\Phi_\varepsilon(y) \leqslant 2\Phi_\varepsilon(0) = 2e(K, \varepsilon)$$

for all $\|y\| \leqslant \varepsilon$. Since for $\|y\| \leqslant \varepsilon$

$$-\Phi_\varepsilon(y) = \inf H_\varepsilon(-y) \leqslant \Phi_\varepsilon(-y) \leqslant 2e(K, \varepsilon),$$

Φ_ε is bounded in $B(0, \varepsilon]$.

(b) $\varepsilon = 0$: Now we can assume that IK is a neighborhood of 0 and that there is an $L_0 \in Y'$ with $E_{L_0}(K, 0) < \infty$. For all $y \in IK$ we have

$$\begin{aligned} |\Phi_\varepsilon(y)| &\leqslant |\Phi_\varepsilon(y) - L_0(y)| + |L_0(y)| \\ &\leqslant E_{L_0}(K, 0) + |L_0(y)|. \end{aligned}$$

Thus Φ_ε is bounded in a neighborhood of 0.

The proof of the theorem is now complete.

Finally we remark that the case $\mathbb{K} = \mathbb{C}$ can be reduced to the case $\mathbb{K} = \mathbb{R}$, if K is $\mathbb{C}$-balanced, e.g.,

$$|\lambda| \leqslant 1,\ x \in K \text{ implies } \lambda x \in K.$$

This is done by looking at all spaces as $\mathbb{R}$-spaces and exchanging Re U for U. If $L \in Y^*_{\mathbb{R}}$ is an optimal algorithm of this new problem, then

$$L_{\mathbb{C}}(y) = L(y) - iL(iy)$$

is an optimal algorithm of the original problem.

This can be seen in the following way: Since K is $\mathbb{C}$-balanced and $L_{\mathbb{C}}$ is $\mathbb{C}$-linear we get

$$\begin{aligned} e(K, \varepsilon) &= \sup\{|Ux|; x \in K, \|Ix\| \leqslant \varepsilon\} \\ &= \sup\{\text{Re } Ux; x \in K, \|Ix\| \leqslant \varepsilon\} \\ &= e_{\mathbb{R}}(K, \varepsilon) \\ &= E_{\mathbb{R}}(K, \varepsilon) \\ &= \sup\{|Ly - \text{Re } Ux|; x \in K, \|Ix - y\| \leqslant \varepsilon\} \\ &= \sup\{|\text{Re } L_{\mathbb{C}}(y) - \text{Re } Ux|; x \in K, \|Ix - y\| \leqslant \varepsilon\} \\ &= \sup\{|L_{\mathbb{C}}(y) - Ux|; x \in K, \|Ix - y\| \leqslant \varepsilon\} \\ &= E_{L_{\mathbb{C}}}(K, \varepsilon). \end{aligned}$$

Here the suffix $\mathbb{R}$ indicates that $e_{\mathbb{R}}(K, \varepsilon)$ and $E_{\mathbb{R}}(K, \varepsilon)$ are the errors of the real problem. Since

$$e(K, \varepsilon) \leqslant E(K, \varepsilon)$$

(see [2]) we see that $L_{\mathbb{C}}$ is an optimal algorithm. $L_{\mathbb{C}}$ is continuous iff L is continuous.

REFERENCES

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2. C. A. MICCHELLI AND T. J. RIVLIN, A survey of optimal recovery, *in* "Optimal Estimation in Approximation Theory". Plenum, New York, 1976.